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Sequential density

Paul Gartside *, Joseph T.H. Lo, Andrew Marsh [1]

Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, 15260 Pittsburgh, PA, USA

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Abstract

We investigate when the product of spaces is (strongly) sequentially separable. We also determine necessary and sufficient conditions on X, for the space $C_p(X)$ to be (strongly) sequentially separable.

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If X is a space and $A \subseteq X$, then the *sequential closure* of A, denoted by $[A]_{\text{seq}}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be *sequentially dense* if $X = [D]_{\text{seq}}$. Define the *sequential density* of X, denoted by $d_s(X)$, to be the minimal cardinality of a sequentially dense set in X. Call X *sequentially separable* if $d_s(X) = \aleph_0$ and *strongly sequentially separable* if X is separable and *every* countable dense subset of X is sequentially dense. Clearly, every strongly sequentially separable space is sequentially separable, and every sequentially separable space is separable.

In this note we show that the sequential density of a power of a space is either countable or very large; determine when products of sequentially separable spaces are sequentially separable; determine when products of strongly sequentially separable spaces are strongly sequentially separable; and give necessary and sufficient conditions for the space $C_p(X)$ of continuous real valued functions on a space X, with pointwise topology, to be sequentially separable or strongly sequentially separable.

Our results on products of (strongly) sequentially separable spaces continue a line of research started by Tall [15] who showed that under MA $+ \neg$CH, the product of less than

* Corresponding author.
E-mail address: gartside@math.pitt.edu (P. Gartside).
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continuum many sequentially separable spaces is sequentially separable. Tall's result was extended to products of $< \mathfrak{p}$ sequentially separable spaces by Matveev [11], who also introduced the class of strongly sequentially separable spaces. Dow, Matveev and Nyikos showed that the Cantor cube 2^κ is strongly sequentially separable if and only if $\kappa < \mathfrak{p}$. A proof appears in the paper [8] by Hrusak and Steprans, who consider a variety of sequential properties of Cantor cubes.

Recall that a *Frechet* space is a space X such that whenever $x \in \overline{A}$, there is a sequence in A converging to x. Clearly, every first countable space is Frechet. It is also immediate that if X is a separable Frechet space, then it is strongly sequentially separable. We show that for a second countable space X, its function space $C_p(X)$ is strongly sequentially separable if and only if it is Frechet; but this is not the case for non second countable X. A continuous image of sequentially separable space is sequentially separable. Hence *cosmic* spaces—the continuous images of separable metric spaces—are sequentially separable. So, for any separable metric space X (or more generally, cosmic X), $C_p(X)$ is cosmic, and hence sequentially separable. We show that $C_p(X)$ is sequentially separable provided the space X has a coarser separable metric topology closely related to the original topology.

For simplicity, we assume all spaces are Tychonoff. A subset S of the real line is called a *Q-set* if each one of its subsets is a G_δ. The cardinal $\mathfrak{q}$ is the smallest cardinal so that for any $\kappa < \mathfrak{q}$ there is a Q-set of size κ. The cardinal $\mathfrak{p}$ is the smallest cardinal so that there is a collection of $\mathfrak{p}$ many subsets of the natural numbers with the strong finite intersection property but no infinite pseudo-intersection. It is known that $\omega_1 \leqslant \mathfrak{p} \leqslant \mathfrak{q} \leqslant 2^{\aleph_0}$, that it is consistent that $\mathfrak{p} < \mathfrak{q}$, and that the cofinality of $\mathfrak{q}$ is uncountable. (See [2] for more on small cardinals including $\mathfrak{p}$.)

Non-sequentially separable products

Our first result is a striking dichotomy on the sequential density of powers of a space. Call a space *non-trivial* if it contains at least two points.

Theorem 1. *Let X be a non-trivial space. Then for any uncountable cardinal κ, either* $\mathrm{cf}(d_s(X^\kappa)) = \aleph_0$ *or* $\mathrm{cf}(d_s(X^\kappa)) > \kappa$.

Proof. Suppose $d_s(X^\kappa) = \mu$ and $\mathrm{cf}(\mu) = \lambda$, with $\aleph_0 < \lambda \leqslant \kappa$. Let $\mu = \bigcup\{\mu_\alpha\colon \alpha \in \lambda\}$, where for each $\alpha \in \lambda$, $\mu_\alpha \in \mu$. Let κ have a partition $(\Gamma_\alpha)_{\alpha\in\lambda}$, where for each $\alpha \in \lambda$, $|\Gamma_\alpha| = \kappa$. Let $D = \{f_\alpha\colon \alpha \in \mu\} \subseteq X^\kappa$ be a sequentially dense subset. For each $\alpha \in \lambda$, let $D_\alpha = \{f_\beta \restriction \Gamma_\alpha\colon \beta \leqslant \mu_\alpha\}$. Note that since $|D_\alpha| < \mu$, there is a $g_\alpha \in X^{\Gamma_\alpha}$ such that $g_\alpha \notin [D_\alpha]_{\mathrm{seq}}$. Define $g = \bigcup_{\alpha\in\kappa} g_\alpha \in X^\kappa$, and note that $g \notin [D]_{\mathrm{seq}}$. $\square$

Theorem 2. *It is consistent that for all $n \in \omega$,*

$$\aleph_1 < d_s\left(2^{\omega_n}\right) = \aleph_\omega < \aleph_{\omega+1} = 2^{\aleph_0} = 2^{\aleph_1}.$$

Proof. Take the model produced by forcing with $\mathrm{Fn}(\omega_\omega, 2)$ over a model V of **GCH**. Let $n \in \omega$. In this model, there is a $D \in [\mathcal{P}(\omega_n)]^{\aleph_\omega}$ such that each $A \subseteq \omega_n$ is a countable union of elements of D. (See [6, Theorem 13(3)] for details, and a related example.) Clearly then in this model, $d_s(2^{\omega_n}) \leqslant \aleph_\omega$. Moreover, $2^{\aleph_0} = 2^{\aleph_1} = \aleph_{\omega+1}$ holds.

We need to show that $d_s(2^{\omega_n}) = \aleph_\omega$ in this model. Suppose $\dot{F}$ is a nice name for a function from $\lambda < \aleph_\omega$ to 2^{ω_n} (considered as a function from $\lambda \times \omega$ to 2), involving only ordinals in $E \subseteq \omega_\omega$, where $|E| = \lambda\omega$. Let $B = \{\beta_\alpha\colon \alpha \in \omega_n\} \subseteq \omega_\omega$ be a subset that misses E.

Let $\dot{H}$ be the generic function. We claim that the element R of 2^{ω_n} obtained by restricting H to B is not in the sequential closure of $\mathrm{ran}(F)$. Suppose $\dot{K}$ is a nice name for a function from ω to λ. This depends only on a countable set $E' \subseteq \omega_\omega$. The model produced by forcing with $\mathrm{Fn}(\omega_\omega, 2)$ over a model V of **GCH** is the same as forcing with $\mathrm{Fn}(E \cup E', 2)$ over V to get $V[G_1]$, and then with $\mathrm{Fn}(\omega_\omega \setminus (E \cup E'), 2)$ to get $V[G_1][G_2]$.

Note that both F and K exist in $V[G_1]$, while R is not in $V[G_1]$. Since the statement that "$F(\mathrm{ran}(K))$ converges" is Δ_0, we can deduce that in $V[G_1][G_2]$, $F(\mathrm{ran}(K))$ does not converge to R. $\square$

Sequentially separable products

Start with the special case of Cantor cubes, 2^κ.

Proposition 3. *For a cardinal κ, the following are equivalent.*

(i) $d_s(2^\kappa) \leqslant \aleph_0$.
(ii) *There is a countable basis $\mathcal{B}$ of a separable metrizable topology on κ such that whenever A is a subset of κ, there is a sequence $(B_n)_{n\in\omega} \subseteq \mathcal{B}$ such that (†) for all $x \in A$ (respectively $x \notin A$) there is an integer N such that for all $n \geqslant N$, $x \in B_n$ (respectively $x \notin B_n$).*
(iii) $\kappa < \mathfrak{q}$.

If a basis $\mathcal{B}$ satisfies (ii), then it is said to be a (†)*basis*. If a sequence of sets (B_n) and a subset A satisfies (†), then the sequence is said to (†)*converge* to A.

Proof. The equivalence of (i) and (ii) is a consequence of identifying the set 2^κ to be the set of all subsets of κ. The topology is the coarsest on κ such that all the elements of the sequentially dense set are continuous. This being separable and metrizable comes from the fact that the basis separates distinct points and is zero-dimensional.

Next, we prove there is a separable metrizable topology τ on κ with a countable (†)basis $\mathcal{B}$ if and only if (κ, τ) is a Q-set. Suppose $X = (\kappa, \tau)$ is a Q-set. We can embed X as a subspace of 2^ω and construct a generalized tangent-disk space T in the following manner. The underlying set T is $(2^\omega \times \{\omega\}) \cup (X \times \omega)$. Let $B_\theta = \{f \in 2^\omega\colon \theta \subseteq f\}$ whenever $\theta \in 2^{<\omega}$, and $U_{f,N} = \{(f,\omega)\} \cup \bigcup_{n\geqslant N} B_{f\restriction n} \times \{n\}$ whenever $f \in X$. The basic open sets of T are of the form $B_\theta \times \{n\}$ (for $\theta \in 2^{<\omega}$ and $n \in \omega$) and $U_{f,N}$ (for $f \in X$ and $N \in \omega$). Our $\mathcal{B}$ shall be the collection of all B_θ together with all their finite unions. Clearly this is a countable basis for 2^ω.

We now prove the equivalence of (ii) and (iii). Just as the tangent-disk space on a Q-set is normal, it can be proved that T is normal. Let A be a subset of X. Then there are disjoint open U and V in T such that $A \times \{\omega\} \subseteq V$ and $(X \setminus A) \times \{\omega\} \subseteq U$. We may assume that both U and V are unions of basic neighborhoods of points of $A \times \{\omega\}$ or of $(X \setminus A) \times \{\omega\}$. Let $V_n = \{f \in 2^\omega\colon\ (f, n) \in V\}$. Then each V_n is the *finite* union of basic open sets, and therefore is a member of $\mathcal{B}$. The sequence (V_n) then (†)converges to A.

Conversely, suppose $X = (\kappa, \tau)$ is a separable metrizable with a countable (†)basis $\mathcal{B}$. Let A be a subset of X. Then there is a sequence (B_n) which (†)converges to A. It is clear that A is a G_δ-set of X since $A = \bigcap_{N\in\omega} \bigcup_{n\geqslant N} B_n$. □

We now look at general products of sequentially separable spaces.

Theorem 4 (Matveev and Tall [11]). *The product of less than $\mathfrak{p}$ many sequentially separable spaces is sequentially separable.*

The cardinal $\mathfrak{p}$ cannot be greater than $\mathfrak{q}$, and it is consistent that $\mathfrak{p} < \mathfrak{q}$ (see [4]).

Lemma 5. *Suppose $X = \prod_{\alpha\in\kappa} X_\alpha$ is sequentially separable. If each X_α is non-trivial, then 2^κ is sequentially separable and hence $\kappa < \mathfrak{q}$.*

Proof. Let D be a countable sequentially dense subset of X, and for each $\alpha \in \kappa$, let x_1^α and x_2^α be two distinct elements of α. For each α, there is a continuous function f_α from X_α to the unit interval I such that $f_\alpha(x_1^\alpha) = 0$ and $f_\alpha(x_2^\alpha) = 1$. As D is countable, there is $\xi_\alpha \in (0, 1)$ such that $f_\alpha(d(\alpha)) \neq \xi_\alpha$ for all $d \in D$. We can assume that $\xi = \xi_\alpha$ for all α.

Then it is clear that D is also sequentially dense in the space $Y = \prod_{\alpha\in\kappa}(X_\alpha \setminus (f_\alpha^{-1}(\xi))$ from which it is easy to see that there is a countable sequentially dense subset of 2^κ. □

Theorem 6. *The product of κ many non-trivial cosmic spaces is sequentially separable if and only if $\kappa < \mathfrak{q}$.*

Proof. The necessity of the condition $\kappa < \mathfrak{q}$ is immediate from Lemma 5.

Let X be a set with $|X| < \mathfrak{q}$. For each $x \in X$ let (Y_x, σ_x) be cosmic. Let $\mathcal{N}_x$ be a countable network for (Y_x, σ_x). Define a new finer topology τ_x on Y_x by letting $\mathcal{N}_x$ be a base of closed and open sets. If we can construct some countable sequentially dense subset of $\prod_{x\in X}(Y_x, \tau_x)$ then we will be done since the identity $i : \prod_{x\in X}(Y_x, \tau_x) \to \prod_{x\in X}(Y_x, \sigma_x)$ is continuous.

(Y_x, τ_x) is second countable, T_2 and zero-dimensional and so it is homeomorphic to a subset of ω^ω. Then for each $x \in X$ the space (Y_x, τ_x) will certainly have a network

$$\mathcal{U}^x = \left\{Y_{i,j}^x\colon\ i \in \mathbb{N},\ j = 1, \ldots, 2^i\right\}$$

such that for fixed i the collection $\{Y_{i,j}^x\colon\ j = 1, \ldots, 2^i\}$ partitions Y_x (i.e., $Y_{i,j_1}^x \cap Y_{i,j_2}^x = \emptyset$ when $j_1 \neq j_2$ and $\bigcup\{Y_{i,j}^x\colon\ j \leqslant 2^i\} = Y^x$). In case Y_x has isolated points we allow some of the $Y_{i,j}^x$'s to be empty. We also require that $Y_{i,j}^x = Y_{i+1,2j-1}^x \cup Y_{i+1,2j}^x$ so that as i increases we get finer partitions of Y^x. For each $x \in X$ choose an arbitrary point of Y_x and label this 0_x. For each non-empty $Y_{i,j}^x$ choose a $y_{i,j}^x \in Y_{i,j}^x$.

$|X| < \mathfrak{q}$ and so X has a (†)basis $\mathcal{B} = \{B_n\colon n \in \omega\}$. Assume that $X \in \mathcal{B}$, $\mathcal{B}$ is closed under finite unions and finite intersections and $B_n \setminus B_m \in \mathcal{B}$ when $n \neq m$.

We define a countable set $F \subset \prod_{x\in X}(Y_x, \tau_x)$ by insisting that $f \in F$ if and only if there exists some collection of pairwise disjoint sets, $\{B_{n_s}\colon s = 0, \ldots, m\}$, from $\mathcal{B}$ and some collection of indices $\{(i_s, j_s)\colon s = 0, \ldots, m\}$ such that $f(x) = y^x_{i_s, j_s}$ for all $x \in B_{n_s}$, and $f(x) = 0_x$ otherwise. We claim that F is sequentially dense in $\prod_{x\in X} Y_x$.

Fix $f \in \prod_{x\in X} Y_x$. For all i, j we define $X(i, j) = \{x \in X\colon f(x) \in Y^x_{i,j}\}$. Note that $\{X(i, j)\colon i \in \mathbb{N},\ j \leqslant 2^i\}$ has the same partitioning properties as each $\{Y^x_{i,j}\colon i \in \mathbb{N},\ j \leqslant 2^i\}$. For all of the sets above we have some sequence $\{B^k(i, j)\colon k \in \omega\}$ from $\mathcal{B}$ that (†)converges to $X(i, j)$. (If $X(i, j) = \emptyset$ then let each $B^k(i, j) = \emptyset$.) Now we will inductively construct sequences $\{C^k(i, j)\colon k \in \omega\}$ that (†)converge to $X(i, j)$, such that for all $i \in \mathbb{N}$ we have $C^k(i, j) \cap C^k(i, j') = \emptyset$ when $j \neq j'$, and $C^k(i + 1, 2j - 1) \cup C^k(i + 1, 2j) \subset C^k(i, j)$ when $j \leqslant 2^i$.

Let $C^k(1, 1) = B^k(1, 1)$ and let $C^k(1, 2) = B^k(1, 2) \cap (X \setminus C^k(1, 1))$ for all $k \in \omega$. $\{C^k(1, 2)\colon k \in \omega\}$ (†)converges to $X(1, 2)$ since $\{X \setminus B^k(1, 1)\colon k \in \omega\}$ †converges to $X \setminus X(1, 1) = X(1, 2)$. Also note that $C^k(1, 1) \cap C^k(1, 2) = \emptyset$ for all $k \in \omega$.

For arbitrary $i \in \mathbb{N}$ and $j \leqslant 2^i$ we define

$$C^k(i + 1, 2j - 1) = B^k(i + 1, 2j - 1) \cap C^k(i, j),$$
$$C^k(i + 1, 2j) = B^k(i + 1, 2j) \cap C^k(i, j) \cap \bigl(X \setminus C^k(i + 1, 2j - 1)\bigr).$$

The fact that $\{C^k(i+1, 2j-1)\}$ is (†)convergent to $X(i+1, 2j-1)$, and $\{C^k(i+1, 2j)\}$ is (†)convergent to $X(i + 1, 2j)$ follows from our inductive hypothesis and the fact that $X(i, j) = X(i + 1, 2j - 1) \cup X(i + 1, 2j)$. It is also clear that $C^k(i + 1, 2j - 1) \cup C^k(i + 1, 2j) \subset C^k(i, j)$. Having constructed all the C^k's at the $i + 1$ level we can see that they are pairwise disjoint as $C^k(i + 1, 2j - 1) \cap C^k(i + 1, 2j) = \emptyset$ and we have assumed that the C^k's are pairwise disjoint at the i level. Note that each $C^k(i, j)$ is in $\mathcal{B}$.

Now we construct a sequence of functions $\{f_k\colon k \in \omega\}$ from F, converging to f. Define f_k by setting $f_k(x) = 0_x$ if $x \notin C^k(i, j)$ for all $i \leqslant k$, all $j \leqslant 2^i$. Otherwise we let $f_k(x) = y^x_{i',j'}$ where $i' = \max\{i \leqslant k\colon x \in C^k(i, j)$ for some $j \leqslant 2^i\}$ and $x \in C^k(i', j')$.

Finally we show that this sequence does in fact converge to f. Let $x \in X$. Let U be an open subset of Y_x such that $f(x) \in U$. Then $f(x) \in Y^x_{i,j} \subset U$ for some i, j. There is some N such that $x \in C^k(i, j)$ for all $k > N$. Then for all $k > \max\{N, i\}$, we know that $f_k(x) = y_{r,s}$ for some r, s with $r \geqslant i$. Also $x \in C^k(r, s) \subset C^k(i, j)$ and so $Y^x_{r,s} \subset Y^x_{i,j}$. This shows that $f_k(x) \in Y^x_{i,j}$. □

Problem 7. Is it true that the product of less than $\mathfrak{q}$ many sequentially separable spaces is sequentially separable?

Strongly sequentially separable products

Now we turn to products of strongly sequentially separable spaces. In general, the product of two regular, T_1, strongly sequentially separable spaces need not be strongly sequentially separable. A consistent example of two strongly sequentially separable topological groups with non strongly sequentially separable product is given in Example 15. However one would expect that such a construction is possible in ZFC, leading to the following problem.

Problem 8. Construct in ZFC two regular, T_1, strongly sequentially separable spaces (or topological groups) whose product is not strongly sequentially separable.

Construct (in ZFC or consistently) a single strongly sequentially separable topological group with non-strongly sequentially separable square.

Restricting the factors to first countable strongly sequentially separable spaces, or separable metric spaces yields positive results.

The following theorem is due to Vaughan [18].

Theorem 9 (Vaughan). *If X is strongly sequentially separable and Y is first countable and separable, then $X \times Y$ is strongly sequentially separable.*

Proof. Let D be countable dense in $X \times Y$, and $(x, y) \in X \times Y$. Let (B_n) be a descending local base at y. Let $D_n = \pi_X(D \cap (X \times B_n))$ for each n. Each D_n is dense in X. We claim that there is a sequence $\{x_n\colon n \in \omega\}$ converging to x satisfying the following: given $N > 0$ there exists K such that for all $i > K$ we have $x_i \in \{D_j\colon\ j > N\}$. If x is isolated this is immediate. So suppose x non-isolated. Pick a maximal family of pairwise disjoint open sets $\{U_n\colon n \in \omega\}$ such that $x \notin \overline{U_n}$ for all n. Then $D = \bigcup_n (D_n \cap U_n)$ is dense, so, by strong sequential separability, there is a sequence (x_n) on D converging to x. Clearly this has the required property.

Now we inductively construct strictly increasing sequences of integers, (n_l) and (i_l), such that $(x_{n_l}, y_{n_l}) \in D \cap (X \times B_{i_l})$. To do the inductive step proceed as follows:

Put $N = i_l$. Pick k so that for all $r \geqslant k$ we have $x_r \in \bigcup\{D_j\colon\ j > i_l\}$. Pick $i > k + n_l$ and put $n_{l+1} = i$. Thus $x_{n_{l+1}} \in \bigcup\{D_j\colon\ j > i_l\}$. Pick j so that $x_{n_{l+1}} \in D_j$ and put $i_{l+1} = j$. Since $x_{n_{l+1}} \in D_j$ there is an $(x_{n_{l+1}}, y') \in D \cap (X \times B_j)$. Put $y_{n_{l+1}} = y'$. Thus $(x_{n_{l+1}}, y_{n_{l+1}}) \in D$.

By construction $\{(x_{n_l}, y_{n_l})\colon\ l \in \omega\}$ converges to (x, y), as required. □

The following theorem is due to Dow, Matveev and Nyikos, and a proof appears in [8] (see also [11]). We extend it to general products of separable metrizable spaces.

Theorem 10 (Dow, Matveev and Nyikos). *The cube 2^κ is strongly sequentially separable if and only if $\kappa < \mathfrak{p}$.*

Theorem 11. *Let X_α be non-trivial separable triable spaces for $\alpha \in \kappa$. Then $X = \prod_{\alpha\in\kappa} X_\alpha$ is strongly sequentially separable if and only if $\kappa < \mathfrak{p}$.*

Proof. First we show that a product of no more than $\kappa < \mathfrak{p}$ many separable metric spaces is strongly sequentially separable. Suppose E is a countable dense set of X (which exists, see [3, p. 81]). Let each X_α have basis $\{B_n^\alpha\colon n \in \omega\}$. Fix $f \in X$. For each α, let $N(f,\alpha) = \{n \in \omega\colon f(\alpha) \in B_n^\alpha\}$. Let

$$\mathcal{F} = \left\{E \cap \pi_\alpha^{-1}(B_n^\alpha)\colon \alpha \in \kappa,\ n \in N(f,\alpha)\right\}.$$

Then this collection $\mathcal{F}$ satisfies the strong finite intersection property and has size $\kappa < \mathfrak{p}$. Therefore it has an infinite pseudo-intersection, $M(f)$. Then $E \cap M(f)$ converges to f.

Now we prove that if $X = \prod_{\alpha\in\kappa} X_\alpha$ is strongly sequentially separable, then 2^κ is strongly sequentially separable and hence $\kappa < \mathfrak{p}$. As X is separable, each X_α must be separable, with countable dense set D_α. Noting that $Y = \prod_{\alpha\in\kappa} D_\alpha$ is also dense in X, Y is strongly sequentially separable. Consider $D_\alpha = C_\alpha^0 \cup C_\alpha^1$, where the C_α^0 and C_α^1 are disjoint non-empty subsets. Fix maps $\phi_\alpha : D_\alpha \to \{0,1\}$, with $\phi_\alpha(x) = i$ if and only if $x \in C_\alpha^i$. Let $\phi = \prod_{\alpha\in A} \phi_\alpha$. We now prove that 2^κ is strongly sequentially separable.

Firstly, 2^κ is separable, since $\kappa \leqslant \mathfrak{c}$ (see [3]).

Suppose now that D is a countable dense set in 2^κ. For each $d \in D$, let D^d be a dense set in $\prod_\alpha C_\alpha^{d(\alpha)}$ (which is just a product of at most continuum many countable—and hence separable—spaces). We claim that $\bigcup_{d\in D} D^d$ is dense in Y.

Let $f \in Y$, and let $U = \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(V_{\alpha_i})$ be a basic open set of Y containing of f. For each α_i choose $d_{\alpha_i} \in V_{\alpha_i}$. Now, choose $d \in D$ such that $d(\alpha_i) = \phi_{\alpha_i}(d_{\alpha_i})$ for all i. As D^d is dense in $\bigcup_{d\in D} D^d$, and each open set V_{α_i} meets $C_{\alpha_i}^{d(\alpha_i)}$, there is an $e \in D^d \cap U$.

Now we notice that $\phi(\bigcup_{d\in D} D^d) = D$, and that $\bigcup_{d\in D} D^d$ is sequentially dense in Y, it must be the case that D is sequentially dense in 2^κ. $\square$

Strong sequential density of $C_p(X)$

From [12, Corollary 4.2.2], we note that $C_p(X)$ is separable if and only if X has a coarser second countable topology. We characterize those spaces X so that $C_p(X)$ is strongly sequentially separable, starting with the special case of X separable metric. The following definition and theorem are relevant. For a proof of Theorem 13 see [7], and more information on the property γ see [7,5].

Definition 12. A family α of subsets of X is called an *ω-cover* of X if for every finite $F \subset X$ there is a $U \in \alpha$ such that $F \subset U$.

Theorem 13 (Gerlits and Nagy). *The following are equivalent*:

(i) *$C_p(X)$ is Frechet*;
(ii) *X has the property γ: for any open ω-cover α of X there is a sequence $\beta \subset \alpha$ such that* $\liminf \beta = X$.

The proof of the following result is similar to the proof of Theorem 13.

Theorem 14. *Let X be T_3 and second countable. Then $C_p(X)$ is strongly sequentially separable if and only if $C_p(X)$ is Frechet.*

Proof. Assume $C_p(X)$ is Frechet. X is second countable so $C_p(X)$ is separable. $C_p(X)$ is Frechet and separable and so $C_p(x)$ is strongly sequentially separable.

Now, to show that $C_p(X)$ strongly sequentially separable implies $C_p(X)$ is Frechet we show that $C_p(X)$ is strongly sequentially separable implies X has the property γ described in Theorem 13.

Let α be an open ω-cover of X. Let $A = \{f \in C_p(X)\colon \overline{f^{-1}(\mathbb{R} \setminus \{0\})} \subset U$ for some $U \in \alpha\}$.

Let $\{B_n\colon n \in \omega\}$ be a countable base for X. If $\overline{B_n} \subset B_m$ then let $f_n^m : X \to [0,1]$ be a continuous function such that $f_n^m(\overline{B_n}) = 1$ and $f_n^m(X \setminus B_m) = 0$. Let B be the linear span over $\mathbb{Q}$ of all such functions. Note that B is countable. We will show that $A \cap B$ is dense in $C_p(X)$.

Let $B(g, x_1, \ldots, x_n; \varepsilon)$ be an arbitrary open set in $C_p(X)$. There is some $U \in \alpha$ such that $\{x_1, \ldots, x_n\} \subset U$. For each $i = 1, \ldots, n$ there are basic open V_i, W_i with $\overline{V_i} \subset W_i \subset \overline{W_i} \subset U$ and $W_i \cap W_j = \emptyset$ when $i \neq j$. We know that for each pair V_i, W_i there is some $f_i \in B$ with $f_i(\overline{V_i}) = 1$ and $f_i(X \setminus W_i) = 0$. Let $q_i \in \mathbb{Q}$ satisfy $|g(x_i) - q_i| < \varepsilon$. Then

$$h = \sum_{i=1}^{n} q_i f_i \in B(g, x_1, \ldots, x_n; \varepsilon).$$

Also if x is not in any W_i then $h(x) = 0$ and so $h^{-1}(\mathbb{R} \setminus \{0\}) \subset \bigcup\{W_i\colon i = 1, \ldots, n\}$. This shows that $\overline{h^{-1}(\mathbb{R} \setminus \{0\})} \subset U$ and so $h \in A \cap B$.

Now $A \cap B$ is countable and dense in $C_p(X)$. Let f^1 denote the constant function at 1. Since $C_p(X)$ is strongly sequentially separable then there is some sequence $\{f_n\colon n \in \omega\}$ in $A \cap B$ (and so in A) converging to f^1. For each $n \in \omega$ we take a $U_n \in \alpha$ for which $\overline{f_n^{-1}(\mathbb{R} \setminus \{0\})} \subset U_n$. Then $\liminf\{U_n\colon n \in \omega\} = X$. To see this note that for any $x \in X$ we have a $n_x \in \omega$ such that $f_n \in B(f^1, x; 1)$ for all $n > n_x$. But this means that $f_n(x) > 0$ and so $x \in U_n$ for all $n > n_x$. $\square$

Todorcevic, in [5], has shown that, consistently, there are two subsets of the reals, X and Y say, with the γ property such that their disjoint sum $X \oplus Y$ does not have the γ property. Since $X \oplus Y$ is T_3 and second countable, and $C_p(X \oplus Y) = C_p(X) \times C_p(Y)$, we have the following example:

Example 15. (Cons (ZFC)) There are separable metric spaces X and Y so that the topological groups $C_p(X)$ and $C_p(Y)$ are strongly sequentially separable but their product, $C_p(X) \times C_p(Y)$ is not strongly sequentially separable.

We now remove the restriction that X be second countable.

Theorem 16. *The function space $C_p(X)$ is strongly sequentially separable if and only if X has a coarser second countable topology, and every coarser second countable topology for X has the property γ.*

Proof. Assume that $C_p(X)$ is strongly sequentially separable. We know that X has a coarser second countable topology. If τ is an arbitrary coarser second countable topology for X then we know that $C_p((X,\tau))$ embeds densely into $C_p(X)$. Also $C_p((X,\tau))$ is separable and so $C_p((X,\tau))$ is strongly sequentially separable. So Theorems 13 and 14 imply that (X,τ) has property γ.

Assume that X has a coarser second countable topology. Then $C_p(X)$ is separable. Let A be a countable dense subset of $C_p(X)$. We wish to show that for any $f \in C_p(X)$ there is some sequence $\{f_i\colon i \in \omega\} \subset A$ such that f is the limit of the sequence. Let $\{U_j\colon j \in \omega\}$ be a base for $\mathbb{R}$ and let $\{V_k\colon k \in \omega\}$ be the collection of all preimages of the U_j's under the functions in $A \cup \{f\}$. This collection is a base for a T_3 topology τ, on X (given $V_k = g^{-1}(U_j)$ we know that there is a $U_{j'}$ with $\overline{U_{j'}} \subset U_j$ and so $\overline{g^{-1}(U_{j'})} \subset U_j$) and clearly each function in $A \cup \{f\}$ is continuous with respect to τ. We know that $C_p(X,\tau)$ is Frechet and so since $f \in \overline{A}$ then we know that f is the limit of some sequence $\{f_i\colon i \in \omega\} \subset A$. $\square$

It is consistent and independent for arbitrary X, that $C_p(X)$ is strongly sequentially separable if and only if $C_p(X)$ is Frechet. In fact it is consistent with ZFC that:

Corollary 17. (Cons (ZFC)) *The following are equivalent*:

(i) $C_p(X)$ *is strongly sequentially separable,*
(ii) X *is countable,*
(iii) $C_p(X)$ *is separable and Frechet.*

Proof. A space X has the property C'' if for any sequence $\{G_n\colon n \in \omega\}$ of open covers of X there is some $U_n \in G_n$ for all $n \in \omega$ such that $\bigcup\{U_n\colon n \in \omega\} = X$. If X has property γ then it has property C''. It is consistent with ZFC that the only subsets of $\mathbb{R}$ with C'' are countable. If a space X has a coarser second countable topology τ with (X,τ) having γ then $\operatorname{ind}(X,\tau) = 0$ and (X,τ) has C'', which implies (X,τ) is homeomorphic to a subset of $\mathbb{R}$ with property C''. Hence the corollary. $\square$

Example 18. Assume that $\omega_1 < \mathfrak{p}$. There is a space X such that $C_p(X)$ is strongly sequentially separable but not Frechet.

Proof. We simply take X to be ω_1 with the discrete topology. This has a coarser second-countable topology and so $C_p(X)$ is a separable, dense subspace of $\mathbb{R}^X$. We know from Theorem 11 that $\mathbb{R}^X$ must be strongly sequentially separable and so $C_p(X)$ is strongly sequentially separable. However $\mathbb{R}^{\omega_1}$ is not Frechet. $\square$

This example leads to the following question.

Problem 19. Is there a consistent example of a space X, such that $C_p(X)$ is strongly sequentially separable, not Frechet and $\mathbb{R}^X$ is not strongly sequentially separable?

The argument of Example 18 shows that if $\kappa < \mathfrak{p}$ then any subset of $\mathbb{R}$ of size κ has the property γ. The converse is also true (folklore) which rules out finding a solution to

Problem 19 by forcing a model of ZFC in which all $\aleph_1$ sized subsets of $\mathbb{R}$ have the γ property but $\omega_1 = \mathfrak{p}$ (so $\mathbb{R}^{\omega_1}$ is not strongly sequentially separable).

Lemma 20. *Every subset of $\mathbb{R}$ of size κ has property γ if and only if $\kappa < \mathfrak{p}$.*

Sequential density of $C_p(X)$

Necessary and sufficient conditions for $C_p(X)$ to be sequentially separable are given. These depend on *property Γ* which is variation on the property γ introduced in the context of $C_p(X)$ strongly sequentially separable.

A collection $\mathcal{C}$ of subsets of a space (X,τ) has property Γ on (X,τ) if and only if given any finite collection of disjoint cozero subsets $\{O_i\}_{i=1}^n$ there exist sequences $\{C_i^j\colon j \in \omega\}_{i=1}^n$, from $\mathcal{C}$ such that $C_{i'}^j \cap C_i^j = \emptyset$ when $i \neq i'$ and $O_i \subset \liminf C_i^j$.

Theorem 21. *$C_p((X,\tau))$ is sequentially separable if and only if there exists a coarser second countable topology μ for X and there exists some collection of μ-closed sets $\mathcal{C} = \{C_i\colon i \in \omega\}$ such that $\mathcal{C}$ has property Γ on (X,τ).*

Proof. Assume that $C_p((X,\tau))$ is sequentially separable, with F a countable sequentially dense subset. We know that X has a coarser second countable topology, μ, with basis obtained by taking all inverse images under elements of F of rational intervals in $\mathbb{R}$. Let $\{O_i\}_{i=1}^n$ be a disjoint collection of τ-cozero sets. Let f be a τ-continuous positive function satisfying: $i-1 < f(x) < i$ for all $x \in O_i$ when $i = 1,\ldots,n$. There is a sequence $\{f_j\colon j \in \omega\}$ from F such that $f_j \to f$. Define $C_i^j = f_j^{-1}([i-1+\frac{1}{j+2}, i-\frac{1}{j+2}])$. We can see that $O_i \subset \liminf C_i^j$ and that $C_{i'}^j \cap C_i^j = \emptyset$ when $i \neq i'$. We can now clearly construct the required countable collection, $\mathcal{C}$ of μ-closed sets with property Γ.

Assume that there exists a coarser second countable topology μ for X and there exists some collection of μ-closed sets $\mathcal{C} = \{C_i\colon i \in \omega\}$ such that $\mathcal{C}$ has property Γ on (X,τ). For convenience we will assume that for all $f \in C_p((X,\tau))$ we have that $f(X) \subset (0,1)$. Let F' be a countable subset of $C_p(X)$ such that given any finite collection $\{C_1,\ldots,C_n\}$ of pairwise disjoint sets from $\mathcal{C}$ and rationals $\{r_1,\ldots,r_n\}$ there is some $f \in F'$ such that $f(x) = r_i$ when $x \in C_i$ and $\min\{r_1,\ldots,r_n\} \leqslant f(x) \leqslant \max\{r_1,\ldots,r_n\}$. Also assume that given $f, g \in F'$ then the functions $\max\{f,g\}$ and $\min\{f,g\}$ are also in F', where $\max\{f,g\}(x) = \max\{f(x),g(x)\}$ for all $x \in X$ and $\min\{f,g\}(x) = \min\{f(x),g(x)\}$ for all $x \in X$. Finally assume that F' is closed under finite rational linear combinations. Now it will suffice to prove that $F = F' \cap C_p(X,(0,1))$ is sequentially dense in $C_p(X,(0,1))$.

Fix $f \in C_p(X,(0,1)$ and an integer $p > 1$. We claim that there exists some sequence $\{f_i^p\colon i \in \omega\}$ from F that converges to f except at $f^{-1}(A_p)$ where $A_p = \{\frac{i}{p^j}\colon j \in \mathbb{N}, i = 1,\ldots,p^j-1\}$. For all $j \in \mathbb{N}$ and $i = 1,\ldots,p^j-1$ define $O_{j,i} = f^{-1}((\frac{i}{p^j}, \frac{i+1}{p^j}))$. By assumption, for fixed j we have sequences $\{B_{j,i}^k\colon k \in \omega\}$ from $\mathcal{C}$ such that $B_{j,i'}^k \cap B_{j,i}^k = \emptyset$ when $i \neq i'$ and $O_{j,i} \subset \liminf B_{j,i}^k$. Let $C_{1,i}^k = B_{1,i}^k$ for relevant i. For each $i \leqslant p^{j+1}$

define $C^k_{j+1,i} = B^k_{j+1,i} \cap C^k_{j,i'}$ when $O_{j+1,i} \subset O_{j,i'}$ (which can clearly only happen for one such i'). Define

$$f^p_k = \frac{1}{2p^k} + \sum_{j\leqslant k}\sum_{i<p^j} g^k_{j,i}$$

where $g^k_{j,i}(x) = \frac{i \bmod p}{p^j}$ for $x \in C^k_{j,i}$ and $g^k_{j,i}(x) \leqslant \frac{p-1}{p^j}$ for all $x \in X$. Note that for all $k \in \omega$ we have $f^p_k \in F$. Then $\{f^p_k\colon k \in \omega\}$ converges to f except at $f^{-1}(A_p)$.

To prove convergence: fix $x \in X \setminus f^{-1}(A_p)$ and $\varepsilon > 0$. There are i_1, j_1 such that $f(x) \in (\frac{i_1}{p^{j_1}}, \frac{i_1+1}{p^{j_1}}) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$. There is some $N > 0$ such that $x \in C^k_{j_1,i_1}$ for all $k > N$. So it suffices to show that if $x \in C^k_{j_1,i_1}$ and $k > j_1$ then $f^p_k(x) \in (\frac{i_1}{p^{j_1}}, \frac{i_1+1}{p^{j_1}})$. To do this we split the sum $\sum_{j\leqslant k}\sum_{i<2^j} g^k_{j,i}$ into two bits.

First we claim that $\sum_{j\leqslant j_1}\sum_{i<2^j} g^k_{j,i} = \frac{i_1}{p^{j_1}}$. We show this by induction on j_1. This is clearly true when $j_1 = 1$. If $x \in C^k_{j_1+1,i_1}$ and $x \in C^k_{j_1,i_2}$ then

$$\sum_{j\leqslant j_1+1}\sum_{i<2^j} g^k_{j,i} = \frac{i_2}{p^{j_1}} + \frac{i_1 \bmod p}{p^{j_1+1}} = \frac{pi_2 + i_1 \bmod p}{p^{j_1+1}}.$$

But by the nested construction of the $C^k_{j,i}$'s we must have $pi_2 + i_1 \bmod p = i_1$.

Now we also have

$$\sum_{j_1<j\leqslant k}\sum_{i<2^j} g^k_{j,i} \leqslant \sum_{j_1<j\leqslant k}\sum_{i<2^j} \frac{p-1}{p^j} = \frac{1}{p^{j_1}} - \frac{1}{p^k}.$$

Finally we get that

$$f^p_k(x) = \sum_{j\leqslant k}\sum_{i<2^j} g^k_{j,i} + \frac{1}{2p^k} > \frac{i_1}{p^{j_1}}$$

and

$$\sum_{j\leqslant k}\sum_{i<2^j} g^k_{j,i} + \frac{1}{2p^k} \leqslant \frac{i_1}{p^{j_1}} + \frac{1}{p^{j_1}} - \frac{1}{p^k} + \frac{1}{2p^k} = \frac{i_1+1}{p^{j_1}} - \frac{1}{2p^k}$$

and so $f^p_k(x) \in (\frac{i_1}{p^{j_1}}, \frac{i_1+1}{p^{j_1}})$.

Now we can construct $\{f_k\colon k \in \omega\}$ converging to f on all of X. We have $\{f^2_k\colon k \in \omega\}$ converging except on $f^{-1}(A_2)$ and $\{f^3_k\colon k \in \omega\}$ converging except on $f^{-1}(A_3)$. Let $\{f^{2,3}_k\colon k \in \omega\}$ be defined as $f^{2,3}_k(x) = \max\{f^2_k(x), f^3_k(x)\}$ for all $k \in \omega$. This clearly converges on $X \setminus (f^{-1}(A_2) \cup f^{-1}(A_3))$. Define $\{f^{5,7}_k\colon k \in \omega\}$ in the same way. Now let $\{f_k\colon k \in \omega\}$ be defined as $f_k(x) = \min\{f^{2,3}_k(x), f^{5,7}_k(x)\}$. Then $\{f_k\colon k \in \omega\}$ converges to f.

Assume that $\{f_k(x)\colon k \in \omega\}$ does not converge for some $x \in (f^{-1}(A_2) \cup f^{-1}(A_3) \cup f^{-1}(A_5) \cup f^{-1}(A_7))$. Without loss of generality assume $x \in f^{-1}(A_2)$. So there is some $\varepsilon > 0$ such that for all $N > 0$ there exists $k > N$ with $f_k(x) \notin (f(x) - \varepsilon, f(x) + \varepsilon)$.

But $\{f_k^{5,7}(x)\colon\ k \in \omega\}$ converges to $f(x)$ so there is some $N_1 > 0$ such that for all $N > N_1$ there is some $k > N$ with $f_k(x) = f_k^{2,3}(x) \leqslant f(x) - \varepsilon$. Using the fact that $\{f_k^3(x)\colon\ k \in \omega\}$ converges to $f(x)$ and the construction of the $f_k^{2,3}$'s we will get some $N_2 > 0$ such that for all $N > N_2$ there will be $k > N_2$ with $f_k^2(x) = f_k^{2,3}(x) \leqslant f(x) - \varepsilon$ and $f_k^3(x) \in (f(x) - \varepsilon, f(x) + \varepsilon)$. This contradicts our definition of $f_k^{2,3}$. So $\{f_k\colon\ k \in \omega\}$ converges on all of X. $\square$

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